Assignment 3

Q4.

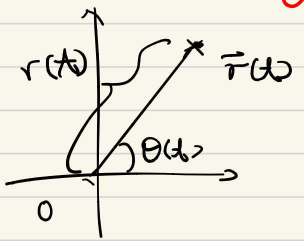
Area swept by the orbit depends only on the length of time interval

Area swept out by the planet in $[t_1, t_2]$.

$$= \frac{1}{2}\int_{\boxed{\alpha}}^{\boxed{\beta}} r(\theta)^2 \, d\theta$$

↳ depends on $t_2 - t_1$

angles



$$= \frac{1}{2}\int_{\theta(t_1)}^{\theta(t_2)} r(t) \, d\theta$$

$r(t)$, $\theta(t)$

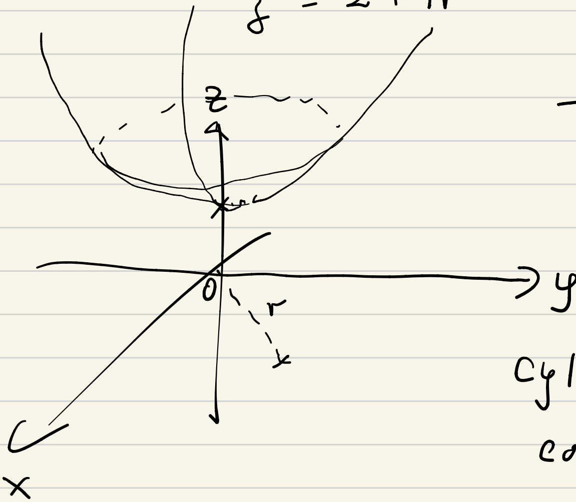
$$= \frac{1}{2}\int_{t_1}^{t_2} r(t)^2 \theta'(t) \, dt$$

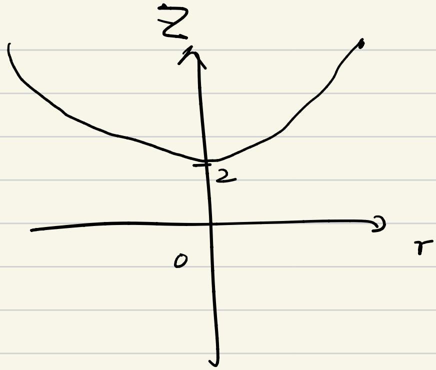
$\theta = \theta(t)$

8(iii)

$$z - 4r^2 = 2$$

$$z = 2 + 4r^2$$

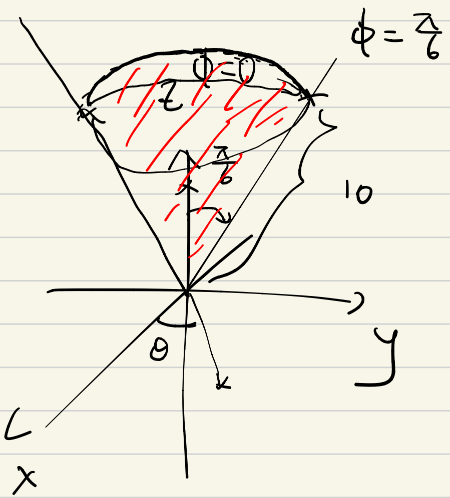




Cylindrical coordinate system

(r, θ, z)

(v)



Spherical coordinate system

(ρ, ϕ, θ)

↑

angles make with positive z-axis

(Monkey Saddle)

Find and classify the critical points of

$f(x,y) = 6xy^2 - 2x^3 - 3y^4$.

(Example in "Differential Multivariable Calculus", by Thomas Kwok-Keung Au)

Problem set 9 (Math 2010 D, 2019-2020)

Q4 (a), (c).

Find the absolute maximum and minimum points of the functions on the given domains

(a) $f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$ on the triangle bounded by the lines $x=0$, $y=2$ and $y=2x$ in the first quadrant.

(c) $f(x,y) = xy$ on the region

$D = \{ (x,y) : x \geq 0, y \geq 0 \text{ and } x^2+y^2 \leq 4 \}$

Sol^n:

(Monkey saddle surface)

$$f(x,y) = 6xy^2 - 2x^3 - 3y^4$$

Set $\begin{cases} \partial_x f = 6y^2 - 6x^2 = 0 \\ \partial_y f = 12xy - 12y^3 = 0 \end{cases}$

$\Leftrightarrow \begin{cases} x = y \quad \text{or} \quad x = -y \\ y = 0 \quad \text{or} \quad x = y^2 \end{cases}$

	$x=y$	$x=-y$
$y=0$	$(0,0)$	$(0,0)$
$x=y^2$	$y=y^2 \Rightarrow \begin{cases}(1,1)\\(0,0)\end{cases}$	$-y=y^2 \Rightarrow \begin{cases}(1,-1)\\(0,0)\end{cases}$

$\therefore$ $(0,0)$, $(1,1)$, $(1,-1)$ are all possible critical points

$\partial_{xx} f = -12x$, $\partial_{xy} f = 12y = \partial_{yx} f$

$\partial_{yy} f = 12x - 36y^2$

Let $D_f(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - (f_{xy}(a,b))^2$

$D_f(0,0) = 0 \implies$ inconclusive (2nd derivative test)

$$\begin{cases} D_f(1,1) = (-12)(-24) - 12^2 = 144 > 0 \\ D_f(1,-1) = (-12)(-24) - 12^2 = 144 > 0 \end{cases}$$

Conclusion by 2nd derivative test :

either local max or local min

$f_{xx}(1,1) = -12 < 0$

$f_{xx}(1,-1) = -12 < 0$

$\therefore$ Local max.

Rmk : We can still explore what type the critical point $(0,0)$ belongs to :

For the plane $x=0$, the surface intersects with this plane and the curve in the yz-plane is

$$z = f(0,y) = -3y^4$$

The curve attains maximum at $y=0$.

This suggests that the critical point $(0,0)$ of $f(x,y)$ can either be a local max or a saddle point.

The surface intersecting with xz plane $(y=0)$ gives the curve

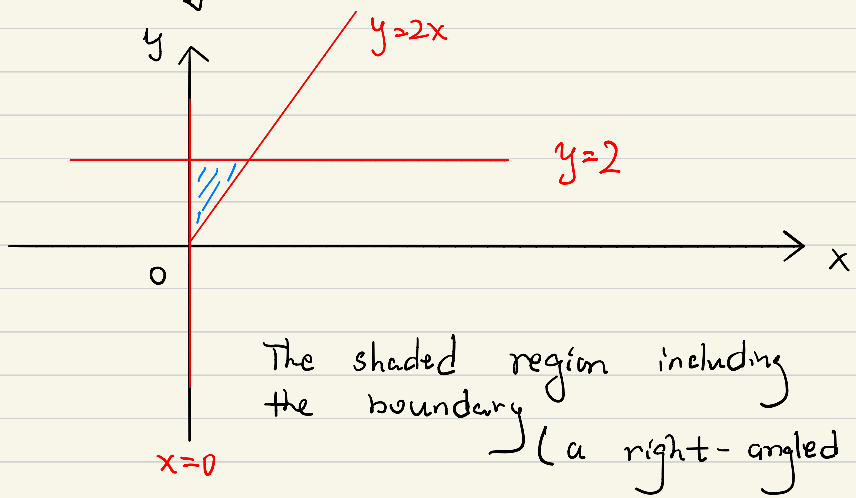
$$z = -2x^3$$

$x=0$ is neither a local max nor a local min of the curve.

This suggests that the critical point $(0,0)$ is a saddle point.

A critical point, but not a local max or local min.

4(a) The region to be considered is



The shaded region including the boundary (a right-angled Δ)

① Interior :

Set $\begin{cases} \dfrac{\partial f}{\partial x} = 4x - 4 = 0 \\ \dfrac{\partial f}{\partial y} = 2y - 4 = 0 \end{cases}$

The solution is $(x,y) = (1,2)$,

which is on the boundary.

This means that No local max, nor local min in the interior

∴ Abs max, Abs min must attain on boundary

② Boundary :

In our example, the boundary composed of three line segments γ_1, γ_2, γ_3

γ_1 : $x=0$, $0 \le y \le 2$

γ_2 : $y=2$, $0 \le x \le 1$

γ_3 : $y=2x$, $0 \le x \le 1$

On γ_1, $f(x,y) = y^2-4y+1$

$= (y-2)^2-3$

attains max at $(0,0)$, $f(0,0)=1$

attains min at $(0,2)$, $f(0,2)=-3$

On γ_2, $f(x,y) = 2x^2-4x-3$

$= 2(x-1)^2-5$

attains max at $(0,2)$, $f(0,2)=-3$

attains min at $(1,2)$, $f(1,2)=-5$

$(y=2x,\ 0\le x\le 1)$

On γ_3, $f(x,y) = 6x^2 - 12x + 1$

$$= 6(x-1)^2 - 5$$

attains max at $(0,0)$, $f(0,0) = 1$

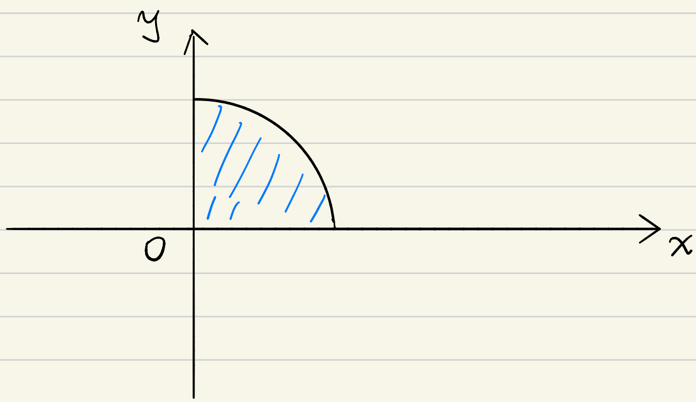
attains min at $(1,2)$, $f(1,2) = -5$

Combining information from these three lines,

f attains abs max at $(0,0)$ $(f(0,0) = 1)$

and abs min at $(1,2)$ $(f(1,2) = -5)$

4(c) The region to be considered is



Same procedures as the example given above, there are no local max, nor local min in the interior.

Boundary :

γ_1 : $x=0$, $0 \le y \le 2$

γ_2 : $y=0$, $0 \le x \le 2$

γ_3 : $x^2+y^2=4$, $0 \le x \le 2$
$0 \le y \le 2$

On γ_1 or γ_2 , $f(x,y) \equiv 0$

On γ_3, $f(x,y) = xy$

$$= \frac{1}{2}\left(x^2+y^2 - (x-y)^2 \right)$$

$$= \frac{1}{2}\left(4 - (x-y)^2 \right)$$

attains max at $x=y=\sqrt{2}$, $f(\sqrt{2},\sqrt{2}) = 2$

attains min when $|x-y|$ is maximized

i.e. $(2,0)$ or $(0,2)$

$f(2,0) = f(0,2) = 0$

Notice that from the definition of the

function and the given domain, it is

easy to observe that the abs min of f is ≥ 0 [$\because$ $x, y \geq 0$]

Conclusion :

The absolute minimum points are

$\{(0,y) : 0 \leq y \leq 2\} \cup \{(x,0) : 0 \leq x \leq 2\}$

The absolute max point is $(\sqrt{2}, \sqrt{2})$

$f(\sqrt{2}, \sqrt{2}) = 2$